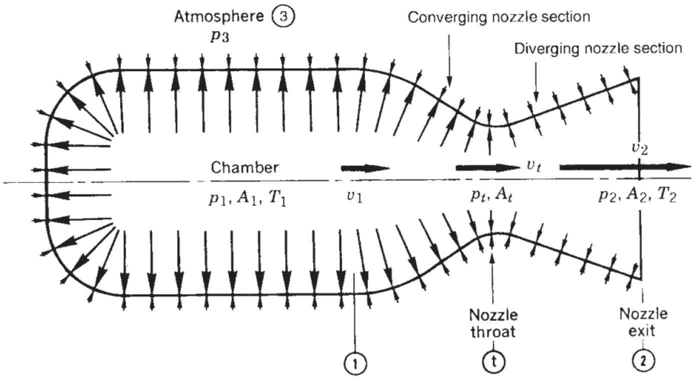
Project Liquid Rocket Equation Reference

Introduction and Constants:

In this sheet the commonly used equations for propulsion related calculations are summarized. Throughout the process it is helpful to keep in mind what variables we are looking for and what variables we can set or know and which are therefore constants. These are various propellant properties like k - ratio of specific heats of gas (property of propellants) and p_1 - chamber pressure (set by us via pressure of feed)
This sheet is derived entirely from the 9^{th} edition of Rocket Propulsion Elements (RPE) (with some additional information supplemented from various sources) and aims to adhere to their naming convention as closely as possible, however, equations are given in SI units without including numerical correction factors like J.
The following figure from ROP will be useful for reference to the variables and subscripts used in the first sections of the summary.



Definitions and Fundamentals

Specific Impulse:
Specific Impulse represents the thrust of the rocket F per unit propellant "weight" flow rate $\dot{w}$. Its general definition is:

$$I_s = \frac{\int_0^t F\,dt}{g_0 \int_0^t \dot{m}\,dt} \tag{1}$$

If we disregard transients and assume mass flow rate $\dot{m}$ to be constant, we can take the derivative of both sides and get:

$$I_s = \frac{F}{\dot{m} \cdot g_0} = \frac{F}{\dot{w}} \tag{2}$$

Characteristic Velocity:
It is used for comparing the relative performance of different chemical rocket propulsion system designs and propellants. As will be seen later c^* is essentially "a constant" independent of nozzle characteristics and can be thought of as related to the efficiency of the combustion process.

$$c^* = \frac{p_1 A_t}{\dot{m}} \tag{3}$$

It relates the quantity $p_1 A_t$ (*whose purpose will be shown later*) to the mass flow rate $\dot{m}$.
General Thrust Equation:

$$F = \dot{m} v_2 + (p_2 - p_3) A_2 \tag{4}$$

The thrust consists of 2 components: *momentum thrust* and *pressure thrust*. When nozzles operate at their *optimum expansion ratio* $p_2 = p_3$ and the pressure thrust is 0.
Effective Exhaust Velocity:
In actual rocket nozzles, the exhaust velocity is not really uniform over the entire exit cross section. A uniform axial velocity c is assumed for all calculations. This effective exhaust velocity c represents an average or mass-equivalent velocity at which propellant is being ejected from the rocket vehicle (*i.e. if we had a perfectly uniform flow and constant mass flow rate, it would be the exhaust velocity for a given thrust*):

$$c = I_s g_0 = \frac{F}{\dot{m}} = v_2 + \frac{(p_2 - p_3) A_2}{\dot{m}} \tag{5}$$

Therefore $F = \dot{m} c$.
Mass Flow Rates:
The OF ratio (*mixture ratio*) r is defined as $r = \frac{\dot{m}_O}{\dot{m}_F}$. Knowing the total mass flow rate of exhaust gases $\dot{m}$, we can determine the individual mass flow rates of oxidizer and fuel:

$$\dot{m}_O = \frac{r}{r+1}\dot{m} \qquad \dot{m}_F = \frac{1}{r+1}\dot{m} \tag{6}$$

Ideal Rocket Propulsion Systems:

The Ideal Rocket Propulsion equations represent an idealization and simplification of the full two- or three-dimensional equations of real aerothermochemical behavior, however, they significantly simplify the calculations involved and "measured actual performances turn out to be usually between 1 and 6% below the calculated ideal values". Here are the main assumptions used (*for full list see ROP p. 46*):

1. All fluid species are gaseous.
2. The working fluid obeys the perfect gas law.
3. There is no heat transfer across any an all gas-enclosure walls; therefore, the flow is adiabatic.
4. The propellant flow rate is steady and constant.
5. The gas velocity, pressure, temperature, and density are all uniform across any section normal to the nozzle axis

These assumptions allow the use of equations, which are detailed below.
Conservation of Mass:
Mass flow rate is conserved, therefore in terms of area A, flow velocity v and specific volume $V = \frac{1}{\rho}$ *(reciprocal of density)* along any cross section:

$$\dot{m} = \frac{Av}{V} = const. \tag{7}$$

Perfect Gas Law:
For perfect gas law we use a modified R value, which is the universal gas constant R' divided by the molecular mass M of the flowing gas mixture:

$$pV = RT \tag{8}$$

Specific Heat Ratio:

$$k = \frac{c_p}{c_v} \qquad c_p = \frac{kR}{k-1} \tag{9}$$

Isentropic Flow Equations
For any isentropic flow process, the following relations may be shown to hold between any two nozzle sections x and y:

$$\frac{T_x}{T_y} = (\frac{p_x}{p_y})^{(k-1)/k} = (\frac{V_y}{V_x})^{k-1} \tag{10}$$

During an isentropic expansion the pressure drops substantially, the absolute temperature drops somewhat less, and the specific volume increases.
Stagnation Conditions:
For ideal gases the enthalpy h can be expressed as the product of the specific heat c_p times the absolute temperature T. Typically, however, we use the concept of *stagnation* or *total* enthalpy. The stagnation enthalpy per unit mass h_0 remains constant in nozzle flows:

$$h_0 = h + \frac{v^2}{2} \tag{11}$$

The concecpt of stagnation enthalpy is essentially equivalent to that of energy conservation - it applied to flows between any two nozzle axial sections gives:

$$h_x - h_y = \frac{1}{2}(v_y^2 - v_x^2) = c_p(T_x - T_y) \tag{12}$$

Based on stagnation enthalpy we can derive other stagnation values, such as stagnation temperature T_0. Since $h = Tc_p$:

$$T_0 = T + \frac{v^2}{2c_p} \tag{13}$$

where T - absolute static fluid temperature (*the one we would measure*). In adiabatic flows the stagnation temperature remains constant (Since T_0 is just $h_0/c_p = const.$). Similarly using 10 a useful pressure ratio between stagnation pressure p_0 and the local pressure p can be defined:

$$\frac{p_0}{p} = (\frac{1 + v^2}{2c_p T})^{k/(k-1)} \tag{14}$$

Important: When local velocities are close to zero, the corresponding local temperatures and pressures approach the stagnation pressure and stagnation temperature. Inside combustion chambers, where gas velocities are typically small, the local combustion pressure essentially equals the stagnation pressure. I.e. in the combustion chamber we have stagnation conditions and $h_0 = h_1$, $T_0 = T_1$, and $p_0 = p_1$.

Mach Number:
Using the velocity of sound $a = \sqrt{kRT}$, we define Mach Number to be:

$$M = \frac{v}{a} = \frac{v}{\sqrt{kRT}} \tag{15}$$

Using Enthalpy Conservation from 12 and isentropic flow from 10, we can now relate stagnation temperature to Mach number:

$$T_0 = T[1 + \frac{1}{2}(k-1)M^2] \qquad M = \sqrt{\frac{2}{k-1}(\frac{T_0}{T} - 1)} \tag{16}$$

Similarly for totally isentropic flows stagnation pressure is:

$$p_0 = p[1 + \frac{1}{2}(k-1)M^2]^{k/(k-1)} \tag{17}$$

Velocity:
Using Enthalpy Conservation 12 nozzle exit velocity v_2 and nozzle inlet velocity v_1 can be related by (assuming no heat losses) $v_2 = \sqrt{2(h_1 - h_2) + v_1^2}$. Using isentropic flow equation 10 (for temperature) to express the enthalpies we get:

$$v_2 = \sqrt{\frac{2k}{k-1} RT_1 [1 - (\frac{p_2}{p_1})^{(k-1)/k}] + v_1^2} \tag{18}$$

This gives us v_2 as a function of the pressure ration $\frac{p_1}{p_2}$. When chamber cross section is large compared to the nozzle throat, the nozzle inlet velocity v_1 is comparatively small and can be neglected. The chamber temperature T_1 is also approximately the stagnation temperature T_0, hence we arrive at:

$$v_2 = \sqrt{\frac{2k}{k-1} \frac{R' T_0}{M} [1 - (\frac{p_2}{p_1})^{(k-1)/k}]} \tag{19}$$

where we also used $R = \frac{R'}{M}$. We see that v_2 depends significantly on the ratio of chamber temperature T_0 and molar mass M i.e. $\frac{T_0}{M}$ and is a function of $\frac{p_2}{p_1}$. For optimum expansion $p_2 = p_3$ and $v_2 = c$.

Nozzle Flow and Throat Condition:

Nozzle Throat Equations - Critical Values:
The *nozzle expansion area ratio* ϵ is defined as:

$$\epsilon = \frac{A_2}{A_t} \tag{20}$$

At the throat, maximum gas flow rate per unit area occurs and Mach number $M = 1$. A unique pressure ratio exists which using stagnation pressure conservation 14 can be found as:

$$\frac{p_t}{p_1} = [\frac{2}{k+1}]^{k/(k-1)} \tag{21}$$

We call the throat pressure p_t the *critical pressure*. Similarly, we can find values for critical specific volume V_t and temperature T_t from 10 and 16:

$$V_t = V_1[\frac{k+1}{2}][1/(k-1)] \qquad T_t = T_1 \frac{2}{k+1} \tag{22}$$

And finally from our velocity equation 19 using critical pressure ratio from 21 we see that critical velocity v_t (velocity at the throat) is $\sqrt{\frac{2k}{k+1} RT_1} = \sqrt{kRT_t} = a_t$, where a_t is the *local sonic velocity* from 15. If we combine these relations at the throat we can express mass flow (which is constant) from 7:

$$\dot{m} = const. = \frac{A_t v_t}{V_t} = A_t p_1 k \frac{\sqrt{[\frac{2}{k+1}]^{(k+1)/(k-1)}}}{\sqrt{kRT_1}} \tag{23}$$

Thrust and Thrust Coefficient:

If we use the equation for thrust 4 and mass flow continuity 23 for the throat, we can express the thrust as:

$$\frac{A_t v_t}{V_t} v_2 + (p_2 - p_3) A_2 \tag{24}$$

And expressing v_2 from our velocity equation 19 and mass flow rate from 23 we get:

$$F = A_t p_1 \sqrt{[\frac{2}{k+1}]^{(k+1)/(k-1)} \frac{2k^2}{k-1} [1 - (\frac{p_2}{p_1})^{(k-1)/k}]} + (p_2 - p_3) A_2 \tag{25}$$

We see a directly proportional relation of thrust to throat area A_t and an almost proportional relation to chamber pressure p_1. We can therefore simplify this equation by introducing a pseudo constant - the *thrust coefficient* C_F:

$$\begin{aligned} C_F &= \frac{F}{p_1 A_t} = \frac{F}{\dot{m} c^*} \\ &= \sqrt{[\frac{2}{k+1}]^{(k+1)/(k-1)} \frac{2k^2}{k-1} [1 - (\frac{p_2}{p_1})^{(k-1)/k}]} + \frac{p_2 - p_3}{p_1} \frac{A_2}{A_t} \end{aligned} \tag{26}$$

The second component of C_F disappears for optimum expansion. With this we see that:

$$F = C_F A_t p_1 \tag{27}$$

Characteristic Velocity:

We defined characteristic velocity in 3. Here we wish to find an expression for c^* in terms of gas properties. We can use the upper half of 26, namely that $C_F = \frac{F}{\dot{m} c^*}$ (*which is just using the definition from 3*) and the fact that $\frac{F}{\dot{m}} = c$, where c is effective exhaust velocity from 5, to write $c^* = \frac{c}{C_F}$. In an optimum expansion $c = v_2$ and we can use the velocity equation 19 for $c = v_2$ and the thrust coefficient 26 to express c^* as

$$c^* = \frac{p_1 A_t}{\dot{m}} = \frac{c}{C_F} = \frac{\sqrt{kRT_1}}{k\sqrt{[\frac{2}{k+1}]^{(k+1)(k-1)}}} \tag{28}$$

We see that by doing so c^* is not dependent on the pressure ratio $\frac{p_2}{p_1}$ and is only a function of k and T_1. Our first equation 3 can be used to determine c^* experimentally. The final expression of 28, however, is the ideal value of c^* obtained from the ideal assumptions.
If we look at our most general expression for thrust 25, we see that we don't see it's dependence on the mass flow rate $\dot{m}$. Another useful interpretation of c^* can then be that it is a constant that "restores" the dependence of thrust in 25 on the mass flow rate (*as opposed to* $p_1 A_t$). I.e. it says that $p_1 A_t$ and $\dot{m}$ differ only by a constant and that **we can think in terms of either $c^* \dot{m}$ or $p_1 A_t$**. Therefore, we can express the thrust F in the following different forms:

$$F = \dot{m} c = p_1 A_t C_F = \dot{m} c^* C_F \tag{29}$$

Comments on Characteristic Velocity and Thrust Coefficient:

With all the large expressions for c^* and C_F it might get confusing to consider where we actually got them and what meaning they have. Fundamentally, if we retrace our steps, we see that we defined C_F from 25, which in turn we expressed from our velocity equation 19 and mass flow continuity at the throat 23. These 2 equations, though we got from:

1. Enthalpy Conservation 12
2. Isentropic Flow Equations 10
3. Mass Flow Continuity 7
4. The fact that $M = 1$ at the nozzle (*Equations for Critical Values*)

Therefore, since c^* is also derived from C_F and the velocity equation for 19 at optimum expansion, *one could argue that C_F and c^* are just* helpful parameters that either we can measure from experiment and use in the future or just help simplify our equations.

Injector Flow Characteristics:

Injector Flow Characteristics: The general equations for the flow of an incompressible fluid through hydraulic orifices give the volumetric (*volume per unit time*) flow rate Q and mass flow rate $\dot{m}$ as:

$$Q = C_d A \sqrt{2\frac{\Delta p}{\rho}} \qquad \dot{m} = Q\rho = C_d A \sqrt{2\rho \Delta p} \tag{30}$$

where C_d is a dimensionless discharge coefficient, ρ - propellant density, A - total cross sectional area of orifice, Δp - pressure drop across injector.
Mixture Ratio:
Using the previous equations, a given pressure drop across injection orifices establishes a mixture (OF) ratio r:

$$r = \frac{\dot{m}_O}{\dot{m}_F} = \frac{(C_d)_O}{(C_d)_F} \frac{A_O}{A_F} \sqrt{\frac{\rho_O}{\rho_F} \frac{\Delta p_O}{\Delta p_F}} \tag{31}$$

These equations can be used to estimate actual mixture ratios from cold water flow test data, measured hole areas and discharge coefficients.
Injection Velocity:

$$v_d = \frac{Q}{A} = C_d \sqrt{2\frac{\Delta p}{\rho}} \tag{32}$$

Combustion Chamber and Nozzle:

Chamber volume V_c is defined as the volume from injector face up to nozzle throat section:

$$V_c = A_1 L_1 + A_1 L_c \left(1 + \sqrt{\frac{A_t}{A_1}} + \frac{A_t}{A_1}\right) \tag{33}$$

where L_1 - cylindrical thrust chamber length, A_1 - chamber area, and L_c - length of the converging conical frustum.

Characteristic Chamber Length:
A characteristic chamber length L^* is defined as the length that a chamber of the same volume would have if it were a straight tube whose diameter is the nozzle throat diameter:

$$L^* = \frac{V_c}{A_t} \tag{34}$$

Here the volume is all the volume up to the throat area.
Stay Time: The stay time t_s of propellant gases is the average value of time spent by each flow element within the chamber volume:

$$t_s = \frac{V_c}{\dot{m} V_1} \tag{35}$$

where V_1 - average specific volume of propellant gases in the chamber.
Exact contours for the converging nozzle section are not critical because most have effectively no separation losses. Typically the Chamber Area A_c is selected such that it is at least three times the throat area A_t.

Heat Transfer and Cooling:

Heat transfer rates vary within the thrust chamber depending on the axial coordinate. The amount of heat transferred by conduction from chamber gases to walls in rocket thrust chambers may be neglected. The largest portion of the heat is transferred by means of convection along with some attributable to radiation. Heat transfer rates in the chamber also increase with chamber pressure p_1. There are **two main cooling methods**: *Steady-state method* (regenerative cooling, radiation cooling) and *Transient Heat Transfer method* (heat sink cooling, ablative materials). In the Steady-state method heat transfer rates and chamber temperatures reach thermal equilibrium, while in the Transient method, these values change over time. Various equations used to analyse cooling, specifically Steady-state cooling have been outlined below:

Friction Pressure Loss - Darcy–Weisbach Equation:
A cooling passage may be considered to be a straight, hydraulic pipe, and its friction pressure loss Δp can be calculated accordingly as:

$$\Delta p = \frac{1}{2} \rho f_D v^2 \frac{L}{D} \tag{36}$$

where ρ - the coolant mass density, L - the length of coolant passage, D - the equivalent diameter defined in 37 below, v - the average velocity in the cooling passage, and f_D - the (*Darcy*) friction loss coefficient. The friction loss coefficient is a function of Reynolds number (*defined below in 45*) and can be found in tables on hydraulic pipes.

Equivalent Diameter:
The equivalent diameter D_E (*often also called the Hydraulic diameter*) is defined to be:

$$D_E = \frac{4A_f}{P_w} \tag{37}$$

where A_f - the cross-sectional flow area and P_w - the wetted perimeter.
Using this term, one can calculate many things in the same way as for a round tube (*for a round tube the the equivalent diameter is just the diameter $D_E = D$*).

Thrust Chamber Thermal Stresses:
Temperature differentials across wall chambers introduce compressive stresses on the inside and tensile stresses on the outside. For simple cylindrical chamber walls this stress s calculated as:

$$s = \frac{2\lambda E \Delta T}{1 - \nu} \tag{38}$$

where λ - the coefficient of thermal expansion of the wall material, E - its modulus of elasticity, and ν - the Poisson ratio of the wall material.

Heat Conduction:

For heat conduction the heat transferred per unit area per unit time q can be written as:

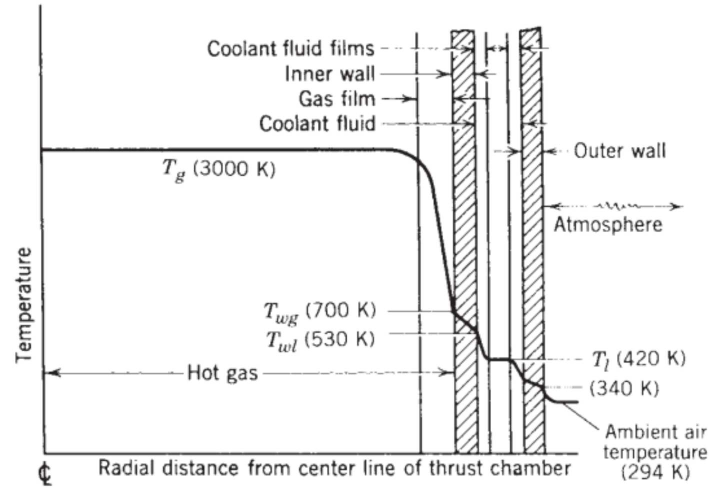
$$q_c = -k \frac{dT}{dL} \approx -k \frac{\Delta T}{t_w} \tag{39}$$

where $\frac{dT}{dL}$ - the temperature gradient, t_w - the wall thickness, and k - the thermal conductivity of the wall material.

Convection Heat Transfer:
The **steady-state heat transfer** through a chamber wall of a liquid-cooled rocket chamber can be treated as a series-resistance-type, steady-state heat transfer problem. A steady-state heat transfer means that:

1. All temperatures in all places remain constant in time
2. All heat flux (*power going through a certain area segment*) is the same going through all the segments (*what goes in, goes out*)

The following figure illustrates the use of subscripts used further below:



Point *2.* of steady-state heat transfer says that the heat flux q going from left to right in the upper figure is the same through all layers. For regeneratively cooled thrust chambers, the steady-state convection heat transfer rate can be expressed in various ways, which are all equal.

$$q = h(T_g - T_l) = \frac{T_g - T_l}{\frac{1}{h_g} + \frac{t_w}{k} + \frac{1}{h_l}} \tag{40}$$

where T_g - the chamber gas temperature, T_l - the coolant liquid temperature, h - an overall film coefficient, which can be expressed from h_g - the gas film coefficient, h_l - the coolant liquid film coefficient, and t_w and k of the chamber wall.

Analogy with Electricity
When analysing steady-state heat transfer, it might be useful to use an analogy to resistances and potentials in electricity. Here Ohm's Law $U = IR$ holds. In this analogy, potential U corresponds to temperature T, current I - to heat flow rate q and, resistance R corresponds to a kind of thermal resistance R'. However, in place of resistance the less familiar electrical conductivity σ is sometimes used. It is just the reciprocal of resistance $R = \frac{1}{\sigma}$. Our analogous variable to electric conductivity σ is thermal conductivity $h = \frac{1}{R'}$.
Therefore, Ohm's Law can be written as $I = \sigma U$ and this analogy gives us some interpretation behind 40. The reciprocals of the film coefficients are just "thermal resistances" and can be summed accordingly to find the overall film coefficient h. Also, we see that the "conductivity" for the wall portion is $\frac{k}{t_w}$.

Convection Heat Transfer through different layers:
Since heat flux through all layers (gas film g, wall w, liquid film l) is the same:

$$q = h_g(T_g - T_{wg}) = \left(\frac{k}{t_w}\right)(T_{wg} - T_w l) = h_l(T_{wl} - T_l) \tag{41}$$

Total Heat Transfer Rate:
Since q varies with the axial distance L within a combustion chamber the total heat transfer per unit time Q can be found by integrating:

$$Q = \int q dA = \pi \int D q dL \tag{42}$$

Film Coefficients - Dittus-Boelter Equation:
The fluid film boundaries established by the combustion products on one side of the wall and by the coolant flow on the other are important quantities controlling the heat transfer across rocket chamber walls. Gas film coefficients establish the amount of the heat transfer rate, and liquid films largely determine the value of the wall temperatures.
Conventional heat transfer theory is usually presented in terms of several dimensionless parameters (*listed in 44, 45, 46 below*). One such preferred relation (which often is modified) inside circular tubes has been the Dittus-Boelter Equation:

$$\frac{h_g D}{k} = 0.023\left(\frac{D v \rho}{\mu}\right)^{0.8} \left(\frac{\mu c_p}{k}\right)^{0.4} \qquad Nu = 0.023 (Re)^{0.8} (Pr)^{0.4} \tag{43}$$

where D - the diameter of the chamber section, v - average local gas velocity, k - conductivity of the gas, μ - the absolute gas viscosity. As can be seen, we can express the gas film coefficient h_g from these equations. The quantities related in 43 are actually known by specific names listed below:

Nusselt Number:

The Nusselt number Nu is the ratio of convective to conductive heat transfer at a boundary in a fluid.

$$Nu = \frac{h_g D}{k} \tag{44}$$

Reynolds Number:

The Reynolds number Re is a dimensionless quantity that helps predict fluid flow patterns in different situations by measuring the ratio between inertial and viscous forces. At low Reynolds numbers, flows tend to be dominated by laminar flow, while at high Reynolds numbers, flows tend to be turbulent.

$$Re = \frac{Dv\rho}{\mu} \tag{45}$$

Prandtl Number:

The Prandtl number Pr is a dimensionless number defined as the ratio of momentum diffusivity to thermal diffusivity. The Prandtl number is dependent only on the fluid and the fluid state. and is often found in property tables for fluids.

$$Pr = \frac{\mu c_p}{k} \tag{46}$$

Improved Gas Film Coefficient Equation - Bartz Equation:

An equation with semiempirical correction factors incorporated by Bartz is:

$$h_g = \frac{0.026}{D^{0.2}}(\frac{c_p \mu^{0.2}}{Pr^{0.6}})(\rho v)^{0.8}(\frac{\rho_{am}}{\rho'})(\frac{\mu_{am}}{\mu_0})^{0.2} \tag{47}$$

where 0 refers to properties at stagnation temperature and am refers to properties at the arithmetic mean temperature of the local free-stream static temperature and the wall temperatures, and ρ' - the free-stream value of the local gas density.

Liquid Film Coefficient:

In steady-state heat transfer analysis, the liquid-film coefficient h_l can be approximated by the following equation:

$$h_l = 0.023\bar{c}\frac{\dot{m}}{A}(\frac{Dv\rho}{\mu})^{-0.2}(\frac{\mu\bar{c}}{k})^{-2/3} \tag{48}$$

where $\dot{m}$ - the liquid fluid mass flow rate, μ - the fluid's absolute viscosity, $\bar{c}$ - its average specific heat, A - the cross-sectional cooling jacket flow area and D the equivalent diameter from 37 of the coolant passage cross section.

Total Heat Rejected to Liquid:

The total heat rejected by the hot gases to the surface of the hot walls and the cooling liquid is:

$$qA = \dot{m}\bar{c}(T_1 - T_2) \tag{49}$$

where T_1 - the initial temperature of the coolant as it enters the cooling jacket and T_2 - its final temperature.